ON ELIGIBILITY BY THE DE BORDA VOTING RULES

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Abstract. We show that a necessary condition for eligibility of a candidate by the set of de Borda's voting rules in [H. Moulin (1988) *Axioms of cooperative decision making*] is not sufficient and we obtain some criterion for eligibility. Let $r(a_i)$ be the score vector of a candidate a_i, R be the set of all vectors $r(a_i)$, and let R' be the Pareto boundary of the convex hull $\operatorname{conv} R$. Then there is a scoring s such that a candidate a wins with respect to the de Borda voting rule β_s if and only if $r(a) \in R'$.

Introduction. Suppose the set A of candidates and the profile u of the voters' preferences are fixed. Let s be a system of scores and β_s be the de Borda rule assigned to s. Further, let $\beta_s(u) \subset A$ be the set of winners w.r.t. this rule. A candidate $a \in A$ is *eligible* w.r.t. the set of de Borda's rules β_s if there is a scoring s such that $a \in \beta_s(u)$. The book [1] suggests the following necessary condition for eligibility of a given candidate a w.r.t. the set of de Borda's rules: a winner a has the score vector $r(a)$ that belongs to the Pareto boundary of the set $R = \{r(a_j),\ a_j \in A\}$; see pages $2-3$ for rigorous definitions.

Unfortunately, the condition in this formulation is not sufficient. On page 5 we give a counter-example using the profile u defined in Eq. (1).

In Theorems 2 and 3 below we prove some version of the eligibility criterion and so we give a complete solution for a problem from [1, p. 249]. The difference between the condition in our criterion and the necessary condition from [1] is that the Pareto boundary of the *convex hull* $\operatorname{conv} R$ must be used instead of the Pareto boundary of the set R itself, here R is the set of all score vectors.

1. Definitions

1.1. **Profiles.** Let $P_1, \ldots, P_n$ be *electors* (or *voters*) and $A = \{a_1; \ldots; a_p\}$ be the set of *candidates* in some elections. Suppose that every elector P_i has an opinion about each candidate such that the candidates are arranged by the strict order $>_i$: the first candidate in this rearrangement is the most favourable for P_i, etc. This strict linear order $>_i$ on A is called the *preference* of the elector P_i and is denoted by u_i. The order u_i is given by the sequence

$$a_{j_1} >_i a_{j_2} >_i \ldots >_i a_{j_p},$$

where $J = (j_1; j_2; \ldots; j_p)$ is a rearrangement of $(1; 2; \ldots; p)$; generally, J depends on the elector P_i.

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In what follows, we write down the elements of the preferences u_i in columns and thus we compose the matrix

$$u = \begin{vmatrix} u_1 \ u_2 \ \dots \ u_n \end{vmatrix}.$$

This matrix is the *profile* of preferences of the voters. Let us denote by $L(A)$ the set of all strict linear orders on A. Then the profiles u are elements of the *space of preferences* $\mathcal{L} = (L(A))^n$.

Suppose that several voters have coinciding preferences u_i, and assume that the order of electors is not important for the voting rules. Then we write down the coinciding columns only once and indicate the respective number of electors in the upper line of the matrix u, e. g.,

$$u = \begin{vmatrix} {\scriptstyle 2} & {\scriptstyle 6} & {\scriptstyle 7} & {\scriptstyle 1} \\ a & b & c & c \\ b & a & a & b \\ c & c & b & a \end{vmatrix}. \tag{1}$$

Notation (1) means that $p = 3$, $n = 16$, and $A = \{a; b; c\}$ is the set of candidates; two electors have the preference $a >_i b >_i c$ $(i = 1, 2)$, six electors have the preference $b >_i a >_i c$ $(i = 3, \dots, 8)$, seven electors have the preference $c >_i a >_i b$ $(i = 9, \dots, 15)$, and the 16th elector has the preference $c >_{16} b >_{16} a$.

We shall use the profile defined by Eq. (1) in Example 2 on page 5 as a counterexample to the erroneous criterion in [1].

Definition 1. Let n and p be positive integers. A *voting rule* φ is a point-to-set map[1] $\varphi\colon \mathcal{L} \to A$ such that $\varphi(u) \neq \varnothing$ for each profile $u \in \mathcal{L}$ of dimension $n \times p$. The candidates $a \in \varphi(u)$ are called *winners* with respect to the rule φ and the profile u.

1.2. **Generalized de Borda's voting rules.** A generalized de Borda's voting rule is defined by the following conditions.

Definition 2. Let $s = (s_0; s_1; \dots; s_{p-1})$ be a vector such that

$$s_0 \le s_1 \le \dots \le s_k \le \dots \le s_{p-1} \text{ and } s_0 < s_{p-1}. \tag{2}$$

The vector s is a *system of scores*, or a *scoring*. Then, each candidate $a \in A$ obtains s_0 points for the last place in the preference of a voter, s_1 points for the last but one place, s_2 points for the third place from the bottom, etc.; clearly, s_{p-1} points are obtained for the top position. The points accumulated by a candidate a from all the voters are summed, and this sum $B_s(a)$ is called the *de Borda estimate* of the candidate a w.r.t. the scoring s, or, briefly, the *s-estimate* for a. The *de Borda voting rule* β_s with the scoring s says that the candidates $a \in A$ who have the maximal sum $B_s(a)$ of points are the *winners* of these elections. Note that the winner may be not unique if two or more candidates $a, b, \dots$ obtain equal (maximal) estimates $B_s(a) = B_s(b) = \dots$. We denote by $\beta_s(u)$ the set of all winners with respect to the profile u.

Example 1. The *standard de Borda voting rule* is based on the system of scores $s = (0; 1; 2; \dots; p-1)$: a candidate obtains 0 points for the last place, 1 point for the last but one place, etc.; a candidate obtains $p-1$ points for the first place. The rule suggests that the winner has the maximal sum of points from all voters.

[1] A *point-to-set map* $f\colon D \to E$ maps each $x \in D$ to some subset $f(x) \subset E$.

The *plurality voting rule* is also a generalized de Borda's rule. The rule corresponds to the scoring s with $s_0 = 0$, $s_1 = 0$, $\ldots$, $s_{p-2} = 0$, $s_{p-1} = 1$. Therefore the sum $B_s(a)$ consists of m units if a is the preferable candidate for m voters; hence the winner by the rule β_s receives the plurality in the preferences of the voters.

The *antiplurality voting rule* is another generalized de Borda's rule. It corresponds to the scoring s such that $s_0 = -1$, $s_1 = 0$, $s_2 = 0$, $\ldots$, $s_{p-1} = 0$. The candidate who wins by the rule β_s has the least number of the last places in the preferences of the electors.

Remark 1. Let β_s be a de Borda's voting rule with the scoring $s = (s_0; s_1; \ldots; s_{p-1})$. If a constant d is added to each score s_k, $k = 0, \ldots, p-1$, then we get the new rule $\beta_{s'}$ with s' such that $s'_k = s_k + d$. Hence all the estimates $B_s(a)$ change to $B_{s'}(a) = B_s(a) + nd$, that is, they acquire the addend nd, which does not depend on a candidate a. Obviously, the sets of winners by the two rules β_s and $\beta_{s'}$ coincide: $\beta_{s'}(u) = \beta_s(u)$ for any profile u. Consequently, the scorings s and s' define *the same* voting rule: $\beta_{s'} = \beta_s$. If we set $d = -s_0$, then we have $s'_0 = 0$. Therefore we can set $s_0 = 0$ for the de Borda rule assigned to any scoring s whenever that is convenient.

Definition 3. Let $F \subset \mathbb{R}^m$ be a closed set. Let $f' \in F$ be a point such that there are no other points $f \in F$ which satisfy the following two conditions

- $f_j \geq f'_j$ for all $j = 1, \ldots, m$, and
- $f_{j_0} > f'_{j_0}$ for some j_0.

The *Pareto boundary* $F' \subset F$ is the subset constituted by all the points f'.

A point $f'' \in F$ belongs to the *weak Pareto boundary* F'' of the set F if there are no points $f \in F$ such that all coordinates of f are greater than the respective coordinates of f''.

Finally, let us recall a helpful statement from the convex analysis.

Proposition 1. *Let F be a bounded convex subset of $\mathbb{R}^m$ and F' be the* (*weak*) *Pareto boundary of F. Then for any point $f^0 \in F'$ there is a linear function $\ell(f) = d_1 f_1 + \cdots + d_m f_m$ with positive* (*resp., nonnegative*) *coefficients d_j such that ℓ achieves its maximum on F at f^0. The converse is also true*: *any point $f^{00} \in F \setminus F'$ is not a point of maximum on F for any linear function with positive* (*resp., nonnegative*) *coefficients.*

2. Eligible and uneligible candidates

Let a profile u of the preferences be fixed. Suppose that the winners are determined by a de Borda's rule β_s from some fixed set $\boldsymbol{\beta}$ of these rules, but we do not know which particular scoring s will be used. Now we want to predict which candidates $a \in A$ can win by a rule $\beta_s \in \boldsymbol{\beta}$ assigned to some scoring s (these candidates are *eligible* w.r.t. this set of rules $\boldsymbol{\beta}$) and which candidates definitely can not win for any $\beta_s \in \boldsymbol{\beta}$ (we call them *uneligible* w.r.t. this set of rules).

We claim that the set of eligible candidates can be found in two important cases (see Theorems 2 and 3) without knowing beforehand the scoring s.

Suppose a profile u is fixed and a is a candidate. The following procedure allows to obtain the *score vector* $r(a)$ for a using the profile u. By construction, the score vector $r(a)$ has $p - 1$ components. The first component $r_1(a)$ equals the number of

electors who regard a as the best. The second component $r_2(a)$ is equal to the number of voters for whom a has either the first or the second place in their preferences; the component $r_3(a)$ is equal to the number of electors having a on some of three first places, etc. Thus we obtain the score vector $r(a) = (r_1(a); r_2(a); \ldots; r_{p-1}(a))$ for every candidate $a \in A$. Obviously, the components of the vector $r(a)$ are non-decreasing:

$$r_1(a) \le r_2(a) \le \cdots \le r_{p-1}(a).$$

Also, we note that the score vector $r(a)$ is found without knowing the scoring s.

Consider the set $R \subset \mathbb{R}^{p-1}$ that consists of p score vectors $r(a)$ for all candidates $a \in A$. Next, find the Pareto boundary R' of the convex hull $\operatorname{conv} R$ of R.

Now we formulate the main result of this article. Theorem 2 contains the criterion of eligibility of a given candidate by de Borda's rules with strict scorings.

Theorem 2. *Suppose a profile u is fixed. Consider the set $\boldsymbol{\beta}$ of all de Borda's rules β_s with the strict scorings s such that*

$$s_0 < s_1 < \cdots < s_{p-1}. \tag{3}$$

Let $a \in A$ be a candidate and $r(a)$ be the score vector for a. Then the candidate a is eligible w.r.t. the set of rules $\boldsymbol{\beta}$ if and only if $r(a) \in R'$, where R' is the Pareto boundary of the convex hull $\operatorname{conv} R$ *of the set $R = \{r(a_j),\ a_j \in A,\ j = 1 \ldots p\}$.*

Proof. According to Remark 1, we suppose that $s_0 = 0$. Let us show that if $a \in \beta_s(u)$ with some scoring s such that inequalities (3) hold, then $r(a) \in R'$. Let the differences between the scores be $d_1 = s_1 - s_0 = s_1$, $d_2 = s_2 - s_1$, $d_3 = s_3 - s_2$, $\ldots$, $d_{p-1} = s_{p-1} - s_{p-2}$. We note that (3) implies $d_{p-k} > 0$ for all $k = 1, 2, \ldots, p-1$.

Now we analyze the summands that contribute to the de Borda estimate $B_s(a)$. The candidate a has the first place in preferences of $r_1(a)$ voters and hence obtains $r_1(a) \cdot s_{p-1}$ points. Next, $r_2(a) - r_1(a)$ voters put the candidate a on the second place, therefore a obtains $(r_2(a) - r_1(a)) \cdot s_{p-2}$ more points. Similarly, for the kth place in the preferences of $(r_k(a) - r_{k-1}(a))$ voters the candidate a obtains $(r_k(a) - r_{k-1}(a)) \cdot s_{p-k}$ points in the sum $B_s(a)$, here $k = 1, 2, \ldots, p-1$. Recall that the last place results in no points since $s_0 = 0$.

By construction, the estimate $B_s(a)$ is the sum of points obtained by a:

$$\begin{aligned}
B_s(a) = r_1(a)s_{p-1} + (r_2(a) - r_1(a))s_{p-2} + (r_3(a) - r_2(a))s_{p-3} + \cdots +\\
+ (r_k(a) - r_{k-1}(a))s_{p-k} + \cdots + (r_{p-1}(a) - r_{p-2}(a))s_1 =\\
= r_1(a)(s_{p-1} - s_{p-2}) + r_2(a)(s_{p-2} - s_{p-3}) + \cdots + r_{p-1}s_1 =\\
= r_1(a)d_{p-1} + r_2(a)d_{p-2} + \cdots + r_{p-1}d_1 = \sum_{k=1}^{p-1} d_{p-k}r_k(a).
\end{aligned} \tag{4}$$

Thus we obtain the formula for the de Borda estimate with the scoring s (see [1, Ch. 9]). For $a \in \beta_s(u)$, the sum in (4) is maximal and the vector $r(a)$ is a point of maximum on R for the linear function $\ell(r) = \sum\limits_{k=1}^{p-1} d_{p-k}r_k$ with positive coefficients d_{p-k}. Moreover, we note that the maximum of a linear function on the convex hull of a set equals the maximum of the function on the set itself. Hence, according to Proposition 1, the

maximum of the function $\ell(r)$ is achieved only at a point of the set R', that is, $r(a) \in R'$ if $a \in \beta_s(u)$.

Conversely, suppose $r(a) \in R' \cap R$. By Proposition 1, we can find a system of positive coefficients d_{p-k} such that the linear function $\ell(r)$ with these coefficients has a maximum on R' (and, consequently, on R) at the point $r(a)$. Using these coefficients, we construct the scoring $s = (s_0; s_1; \dots ; s_{p-1})$ by setting $s_0 = 0$, $s_1 = d_1$, $s_2 = d_1 + d_2$, $s_3 = d_1 + d_2 + d_3$, $\dots$. Then from Eq. (4) it follows that the sum $B_s(a)$ achieves its maximum at $a \in A$ and therefore $a \in \beta_s(u)$. □

Obviously, we can modify slightly the assumptions of Theorem 2 and consider a wider set of de Borda's rules, namely, the set $\boldsymbol{\beta}'$ of rules β_s with the scorings s that satisfy condition (2) but may not satisfy (3). Then, using Proposition 1 again, we obtain

Theorem 3. *Fix a profile u and consider the set $\boldsymbol{\beta}'$ of all de Borda's rules β_s with the scorings s that satisfy condition* (2). *Find the score vectors $r(x)$ for all candidates $x \in A$ and consider the set R that consists of all vectors $r(x)$. Let R'' be the weak Pareto boundary of the convex hull of R. Then any candidate a such that $r(a) \in R''$ is eligible w.r.t. this set of voting rules, and only these candidates are eligible.*

Remark 2. Of course, the conclusions of the two theorems are not true if the set $R \cap R'$ (or $R \cap R''$) is replaced by the Pareto boundary (or the weak Pareto boundary) of the set R itself. (We recall that the set $R \cap R'$ (or $R \cap R''$) consist of the vectors $r(a)$ that belong to the Pareto boundary (or to the weak Pareto boundary) of $\operatorname{conv} R$.) Indeed, the sets $R \cap R'$ and $R \cap R''$ can be more narrow than the (weak) Pareto boundary of R.[2] This situation is illustrated by the following example.

Example 2 (A. V. Shapovalov, private communication)**.** Let $p = 3$ and $n = 16$. Consider profile (1); then we have $r(a) = (2; 15)$, $r(b) = (6; 9)$, $r(c) = (8; 8)$.

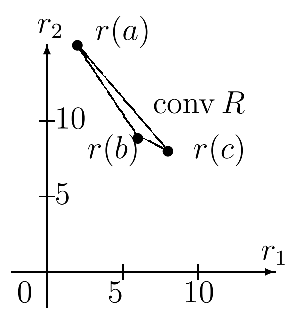

FIGURE 1. Three vectors $r(a)$, $r(b)$, $r(c)$ and their convex hull $\operatorname{conv} R$

The Pareto boundary of the set $R = \{r(a); r(b); r(c)\}$ contains all the three vectors $r(a)$, $r(b)$, and $r(c)$, but only $r(a)$ and $r(c)$ belong to the sets R' and R'' (see Fig. 1).

[2]Clearly, since $R \cap R'$ is a part of the Pareto boundary of R (and $R \cap R''$ is a part of the weak Pareto boundary of R), the condition of [1] is necessary. Namely, any winner a by some rule β_s has the (weak) Pareto optimal (i. e., belonging to the (weak) Pareto boundary of R) score vector $r(a)$. But this condition is not sufficient: not every candidate a having the (weak) Pareto optimal score vector $r(a)$ can win by some voting rule β_s.

Hence only a and c can be the winners by some of the generalized de Borda rules if we have profile (1). The vector $r(b)$ is Pareto optimal, but none of the linear functions $\ell(r_1; r_2) = d_2 r_1 + d_1 r_2$ with positive or nonnegative coefficients d_1 and d_2 achieves its maximum on R at the point $r(b)$. Therefore the candidate b is uneligible neither w.r.t. the set $\boldsymbol{\beta}$ nor w.r.t. the set $\boldsymbol{\beta}'$: there is no rule β_s such that b is a winner.

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Reference

[1] H. Moulin, *Axioms of cooperative decision making.* Cambridge University Press, 1988.

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